A Comparative Study of Equilibrium Equity Premium under Discrete Distributions of Jump Amplitudes

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Abstract

In this paper, we compare equilibrium equity premium under discrete distributions of jump amplitudes. In particular, we consider the binomial and gamma distributions because of their applicability in finance. For the binomial, we assume that the price movement is allowed to either increase or decrease with probability p or $1 - p$ respectively. n is the trading period thereby forming a vector x of jump sizes (shifts) whose distribution is a binomial over time. For the gamma, the jumps are taken to be rare events following a Poisson distribution whose waiting times between them follows a gamma. In both distributions, the optimal consumption of the investor is affected by the deterministic time preference function $y(t)$ but it has no effect on the diffusive and rare-events premia thereby not affecting the equilibrium equity premium. Also, for $n, k = 0$, the volatility effect on the equity premium is the same in both the power and square root utility functions although the equity premium is not affected by the wealth process $V(t)$. However, the wealth process affects the equity premium of the quadratic utility fuction. We observe no significant differences in equity premium for the two discrete distributions.

Keywords

Binomial Distribution, Gamma Distribution, Jump Size, Equity Risk Premium, Jump Diffusion

1. Introduction

The equity risk premium or simply equity premium, the rate by which risky stocks are expected to outperform safe fixed-income investments, such as government bonds and bills, is perhaps the most important index in finance. This is the investor's compensation for taking on the relatively higher risk of the equity market. The equity risk premium is found by subtracting the estimated bond return from the estimated stock return. In our early work, we had considered the impact of utility functions in the production economy with jumps under an arbitrary jump size and derived analytical formulae for an equity premium for the power, exponential, square root and quadratic utility functions. However, we were unable to simulate graphs because of the jump size being arbitrary. In this paper, we derive numerical formulae for an equity premium and simulate graphs by imposing a Binomial distribution on the jump sizes. We then compare the results with those obtained by simulating the Gamma distribution of Jump Amplitudes. Jump diffusion has been widely explored in the area of option pricing but little work has been done to ascertain the behaviour of equity premium under jump diffusion models.

[1]-[4] studied the Pricing of Options under Jump-Diffusion Processes, and derived the appropriate characterization of asset market equilibrium when asset prices follow jump-diffusion process. They developed the general methodology for pricing options on such assets. By imposing specific restrictions on distributions and preferences, [2] formulated a tractable option pricing model that is valid even when jump risk is systematic and non-diversifiable. The dynamic hedging strategies justifying the option pricing model were described and comparisons were made throughout to the analogous problem of pricing options under stochastic volatility.

Jump Diffusion Option Valuation in Discrete Time was proposed by [5] and later developed by [6]-[16]. Multivariate jumps were superimposed on the binomial model of [17] to obtain a model with a limiting jump diffusion process. The model proposed by [5] incorporated the early exercise feature of American options as well as arbitrary jump distributions. The model yielded an efficient computational procedure that can be implemented in practice. To illustrate the model, [5] applied it to characterize the early exercise boundary of American options with certain types of jump distributions.

This paper is related to a number of papers including [11] [18]-[24] solved for the equity premium in an economy with a robust agent that has recursive utility.

2. The Model

This paper is based on theoretical model of [14] and also further elaboration by [25] and [26]. Consider a Jump Diffusion process;

$$\frac{\mathrm{d}X_t}{X_{t^-}} = \mu \mathrm{d}t + \delta \mathrm{d}B_t + \left(\mathrm{e}^x - 1\right)\mathrm{d}N_t - \lambda \mathbf{E}\left(\mathrm{e}^x - 1\right)\mathrm{d}t.$$

The gamma distribution arises naturally when we consider waiting times between Poisson distributed events as relevant. It can be thought of as a waiting time between Poisson distributed events.

The probability density function is the waiting time until the kth Poisson event with a rate of change λ. This is given by

$$\mathbf{P}(x) = \frac{\lambda\left(\lambda x\right)^{k-1}}{\left(k-1\right)!}\mathrm{e}^{-\lambda x}.$$

Now, for $x \sim G(k,\theta)$ where $\theta = \dfrac{1}{\lambda}$, the gamma probability density function is

$$\frac{x^{k-1}\mathrm{e}^{\frac{-x}{\theta}}}{\Gamma(k)\theta^k}$$

where x is a vector of jump amplitudes, k is the number of occurrences of an event and $\left(e = 2.71828\cdots\right)$. In our case, k is the number of times we observe the jumps. We realise that if k is a positive integer, $\Gamma(k) = (k-1)!$ is the gamma function. The value $\theta = \dfrac{1}{\lambda}$ is the mean number of jumps per time unit and λ is the mean time between jumps.

We still subtract the expected value from the drift so that the process becomes more volatile and hence a martingale because its future is unexpected. If we apply Itô Lemma with Jumps we have,

$$\mathrm{d}\ln X_t = \left\{ \left[\mu - \lambda \mathbf{E}\left(\mathrm{e}^x - 1\right) \right] - \frac{1}{2}\delta^2 \right\} \mathrm{d}t + \delta \mathrm{d}B_t + x\mathrm{d}N_t.$$

By integration we have

$$\ln \frac{X_T}{X_t} \equiv Y_\tau = \left[\mu - \lambda \mathbf{E}\left(\mathrm{e}^x - 1\right) - \frac{1}{2}\delta^2 \right]\tau + \delta B_t + \sum_{i=1}^{N_\tau} x_i, \ \text{ for } \tau = T - t$$

where Y_τ is the continuously compounded investment return over the period from time t to T and τ is the investment period.

Suppose also that, at the risk-free rate ρ, the money market account $X_0(t)$ is such that

$$\mathrm{d}X_0(t) = \rho(t) X_0(t)\mathrm{d}t$$

whose total supply is assumed to be zero. Consider here that ρ is risk-free because it is the rate for the money account.

We study comparatively the general equilibriums of one investor who wishes to maximize his expected reward function

$$\max \mathbf{E}_t \int_t^T y(t) U(r_t)\mathrm{d}t,$$

subject to

$$\frac{\mathrm{d}V_t}{V_t} = \left[\rho + \omega\phi - \omega\lambda\mathbf{E}\left(\mathrm{e}^x - 1\right) - \frac{r_t}{V_t} \right]\mathrm{d}t + \omega\delta\mathrm{d}B_t + \omega\left(\mathrm{e}^x - 1\right)\mathrm{d}N_t$$

in an economy with jumps when jump amplitudes follow the binomial and gamma distributions for some time preference function $y(t)$.

3. Results and Discussion

Theorem 1. *If X is a vector of binomially distributed jump sizes, an investor's equilibrium equity premium with CRRA power utility function* $U(r_t) = \dfrac{r_t^\beta}{\beta}, 0 < \beta < 1,$ *in the production economy with jump diffusion is given by*

$$\phi = -\left(\beta - 1\right)\delta^2 + \lambda\left(1 + \left(\mathrm{e} - 1\right)p\right)^n - \lambda\left(1 - p + p\mathrm{e}^\beta\right)^n - \lambda + \lambda\left(1 - p + p\mathrm{e}^{\beta-1}\right)^n$$

where $\phi_\delta = -\left(\beta - 1\right)\delta^2$ is the diffusive risk premium and
$\phi_N = \lambda\left(1 + \left(\mathrm{e} - 1\right)p\right)^n - \lambda\left(1 - p + p\mathrm{e}^\beta\right)^n - \lambda + \lambda\left(1 - p + p\mathrm{e}^{\beta-1}\right)^n$ is the rare-event premium.

Proof. If *X* is a random variable with a binomial distribution, then $Y = \mathrm{e}^X$ is a logbinomial random variable.
In particular, if $X \sim B(n, p)$ and $Y = \mathrm{e}^X$ then $Y^k = \mathrm{e}^{kX}$. Also $\mathbf{E}\left[\mathrm{e}^{kX}\right] = m_X(k)$ where $m_X(k)$ is the moment-generating function of *X* evaluated at *k*. Hence

$$\mathbf{E}\left[\mathrm{e}^{kX}\right] = \left(1 - p + p\mathrm{e}^k\right)^n$$

and so

$$\mathbf{E}\left[\mathrm{e}^X\right] = \left(1 - p + p\mathrm{e}\right)^n = \left(1 + \left(\mathrm{e} - 1\right)p\right)^n = m_X(1).$$

Let $X = x$ be a vector of binomially distributed jump sizes then for the power utility function of [25], the rare-event premium $\phi_N = \lambda E\left[\left(\mathrm{e}^x - 1\right)\left(1 - \left(\mathrm{e}^x\right)^{\beta-1}\right)\right]$ which is $\phi_N = \lambda E\left[\mathrm{e}^x - \mathrm{e}^{x+x(\beta-1)} - 1 + \mathrm{e}^{x(\beta-1)}\right].$

Now

$$\mathbf{E}\left[\mathrm{e}^{x+x(\beta-1)}\right] = \mathbf{E}\left[\mathrm{e}^{x(1+\beta-1)}\right] = \mathbf{E}\left[\mathrm{e}^{\beta x}\right] = \left(1 - p + p\mathrm{e}^\beta\right)^n = m_X(\beta).$$

$$\mathbf{E}\left[e^{x(\beta-1)}\right]=\left(1-p+pe^{\beta-1}\right)^{n}=m_{X}\left(\beta-1\right).$$

Therefore, our rare-event premium

$$\phi_{N}=\lambda\left[\mathbf{E}\left(e^{x}\right)-\mathbf{E}\left(e^{x+x(\beta-1)}\right)-1+\mathbf{E}\left(e^{x(\beta-1)}\right)\right] \quad \text{now becomes}$$

$$\phi_{N}=\lambda\left[\left(1+(e-1)p\right)^{n}-\left(1-p+pe^{\beta}\right)^{n}-1+\left(1-p+pe^{\beta-1}\right)^{n}\right] \quad \text{which implies that our equity premium is now}$$

$$\phi=-(\beta-1)\delta^{2}+\lambda\left(1+(e-1)p\right)^{n}-\lambda\left(1-p+pe^{\beta}\right)^{n}-\lambda+\lambda\left(1-p+pe^{\beta-1}\right)^{n}. \qquad \square$$

The optimal consumption of the investor is affected by the deterministic time preference function $y(t)$ but it has no effect on the diffusive and rare-events premia. In addition, the price of the diffusive risk $\phi_{\delta}=-(\beta-1)\delta^{2}$ is always positive for $0<\beta<1$ and

$$\phi_{N}=\lambda\left(1+(e-1)p\right)^{n}-\lambda\left(1-p+pe^{\beta}\right)^{n}-\lambda+\lambda\left(1-p+pe^{\beta-1}\right)^{n} \quad \text{is the price of the jump risk.}$$

As can be seen in **Figure 1**, for $n=0$, the equity premium is almost zero when volatility is zero. This is consistent with the result for normally distributed jump sizes. Also **Figure 2** shows that, as β approach zero from the right, the equity premium increases significantly and *vice-versa*.

Theorem 2. *For a gamma distribution of jump sizes, an investor's equilibrium equity premium with CRRA power utility function* $U\left(r_{t}\right)=\dfrac{r_{t}^{\beta}}{\beta},0<\beta<1,$ *in the production economy with jump diffusion is given by*

$$\phi=-(\beta-1)\delta^{2}+\frac{\lambda}{\left(1-\frac{1}{\lambda}\right)^{k}}-\frac{\lambda}{\left(1-\frac{\beta}{\lambda}\right)^{k}}-\lambda+\frac{\lambda}{\left(1-\frac{(\beta-1)}{\lambda}\right)^{k}}$$

where $\phi_{\delta}=-(\beta-1)\delta^{2}$ is the diffusive risk premium and $\phi_{N}=\dfrac{\lambda}{\left(1-\frac{1}{\lambda}\right)^{k}}-\dfrac{\lambda}{\left(1-\frac{\beta}{\lambda}\right)^{k}}-\lambda+\dfrac{\lambda}{\left(1-\frac{(\beta-1)}{\lambda}\right)^{k}}$ is
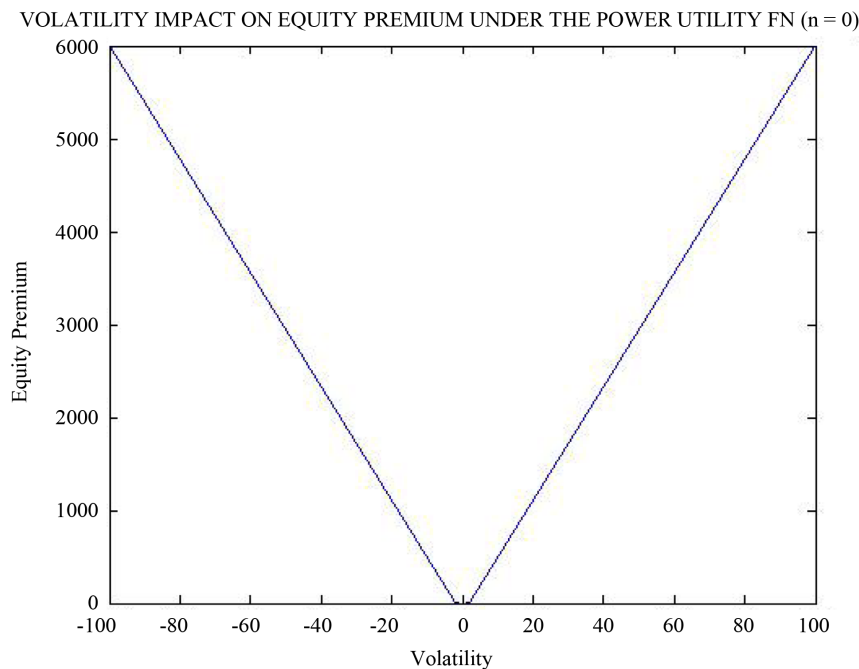
the rare-event premium.

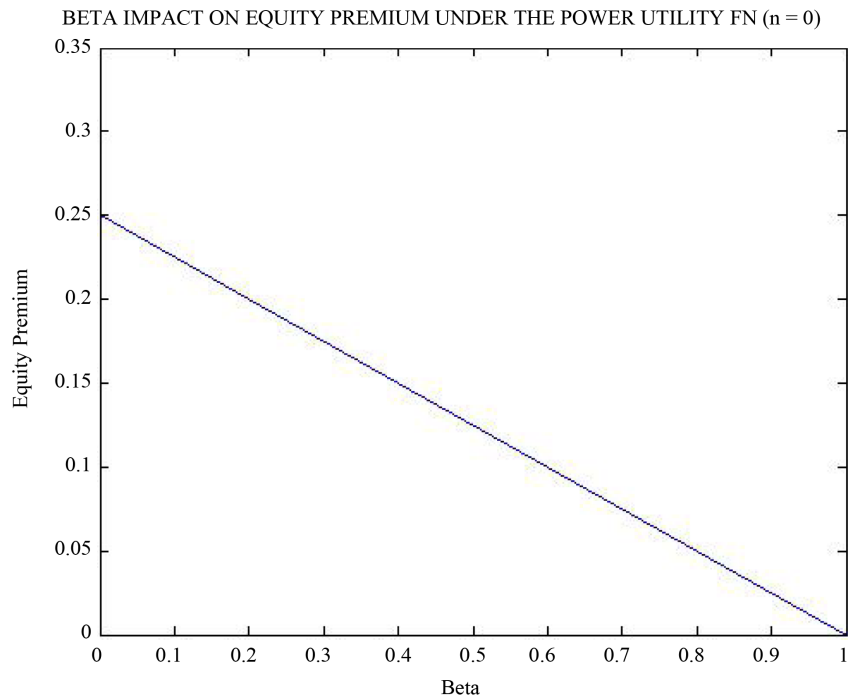Figure 1. Power utility volatility effect under binomial distribution.

BETA IMPACT ON EQUITY PREMIUM UNDER THE POWER UTILITY FN (n = 0)

Figure 2. Power utility beta effect under binomial distribution.

Proof. If x follows a gamma distribution, that is $x \sim G(\lambda, k)$ then $Y = \mathrm{e}^x$ is a log-gamma random variable with parameter

$$\mathbf{E}\left[Y^u\right] = \mathbf{E}\left[\mathrm{e}^{ux}\right] = \frac{1}{\left(1 - \dfrac{u}{\lambda}\right)^k}$$

for some constant u. This is just the moment generating function of x evaluated at u.

For the power utility function, the equilibrium equity premium ϕ was given by

$$\phi = -(\beta-1)\delta^2 + \lambda\left[\mathbf{E}\left(\mathrm{e}^x\right) - \mathbf{E}\left(\mathrm{e}^{x+x(\beta-1)}\right) - \mathbf{E}[1] + \mathbf{E}\left[\mathrm{e}^{x(\beta-1)}\right]\right]$$

where our rare-event premium

$$\phi_N = \lambda\left[\mathbf{E}\left(\mathrm{e}^x\right) - \mathbf{E}\left(\mathrm{e}^{x+x(\beta-1)}\right) - \mathbf{E}[1] + \mathbf{E}\left[\mathrm{e}^{x(\beta-1)}\right]\right]$$

[25].

Now since $x \sim G(\lambda, k)$,

$$\mathbf{E}\left(\mathrm{e}^x\right) = \frac{1}{\left(1 - \dfrac{1}{\lambda}\right)^k} = M_x(1).$$

$$\mathbf{E}\left(\mathrm{e}^{x+x(\beta-1)}\right) = \mathbf{E}\left(\mathrm{e}^{\beta x}\right) = \frac{1}{\left(1 - \dfrac{\beta}{\lambda}\right)^k} = M_x(\beta).$$

$$\mathbf{E}\left(\mathrm{e}^{(\beta-1)x}\right) = \frac{1}{\left(1 - \dfrac{(\beta-1)}{\lambda}\right)^k} = M_x(\beta-1).$$

Therefore our rare-event premium ϕ_N now becomes

$$\phi_N = \frac{\lambda}{\left(1-\dfrac{1}{\lambda}\right)^k} - \frac{\lambda}{\left(1-\dfrac{\beta}{\lambda}\right)^k} - \lambda + \frac{\lambda}{\left(1-\dfrac{(\beta-1)}{\lambda}\right)^k}$$

which implies that our equilibrium equity premium is

$$\phi = -(\beta-1)\delta^2 + \frac{\lambda}{\left(1-\dfrac{1}{\lambda}\right)^k} - \frac{\lambda}{\left(1-\dfrac{\beta}{\lambda}\right)^k} - \lambda + \frac{\lambda}{\left(1-\dfrac{(\beta-1)}{\lambda}\right)^k}. \qquad \Box$$

The optimal consumption of the investor is affected by the deterministic time preference function $y(t)$ but it has no effect on the diffusive and rare-events premia. In addition, the price of the diffusive risk

$\phi_\delta = -(\beta-1)\delta^2$ is always positive for $0 < \beta < 1$ and $\phi_N = \dfrac{\lambda}{\left(1-\dfrac{1}{\lambda}\right)^k} - \dfrac{\lambda}{\left(1-\dfrac{\beta}{\lambda}\right)^k} - \lambda + \dfrac{\lambda}{\left(1-\dfrac{(\beta-1)}{\lambda}\right)^k}$ is the

price of the jump risk.

We realize in **Figure 3** and **Figure 4** that, for $k = n = 0$, the equity premium is almost zero when the volatility is zero and the effect of beta is also the same as in the Binomial distribution respectively.

Theorem 3. *In the production economy with jump diffusion under a vector of binomially distributed jump sizes, the investor's equilibrium equity premium with square root utility function* $U(r_t) = \sqrt{r_t}, r_t > 0$, *is given by*

$$\phi = \frac{1}{2}\delta^2 + \lambda\left(1+(e-1)p\right)^n - \lambda\left(1-p+pe^{\frac{1}{2}}\right)^n - \lambda + \lambda\left(1-p+pe^{-\frac{1}{2}}\right)^n$$
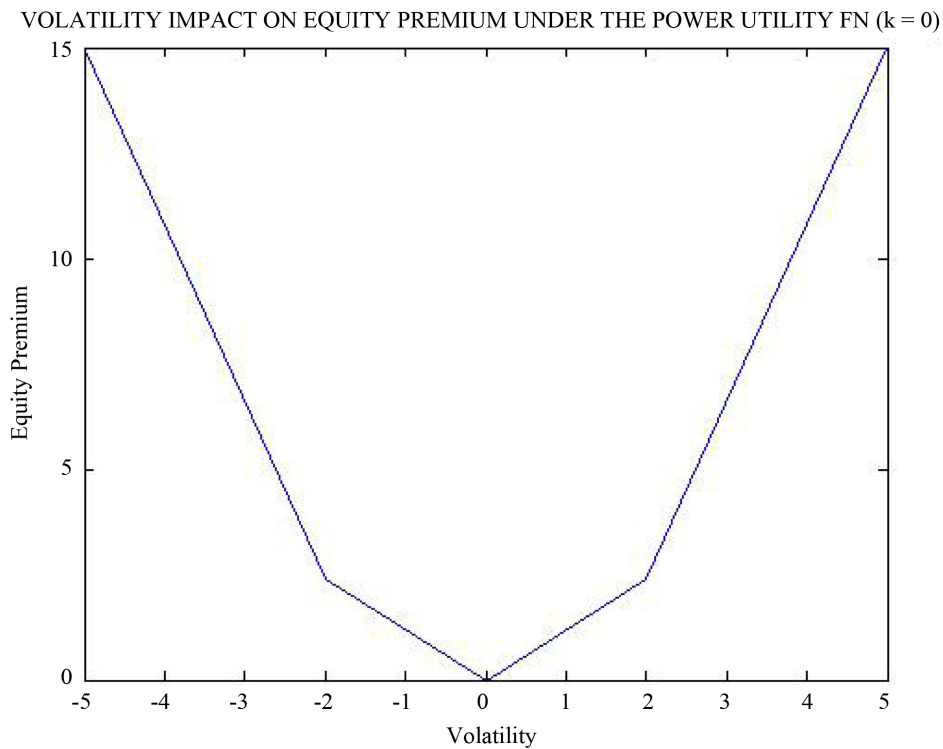
VOLATILITY IMPACT ON EQUITY PREMIUM UNDER THE POWER UTILITY FN (k = 0)

Figure 3. Power utility volatility effect under gamma distribution.

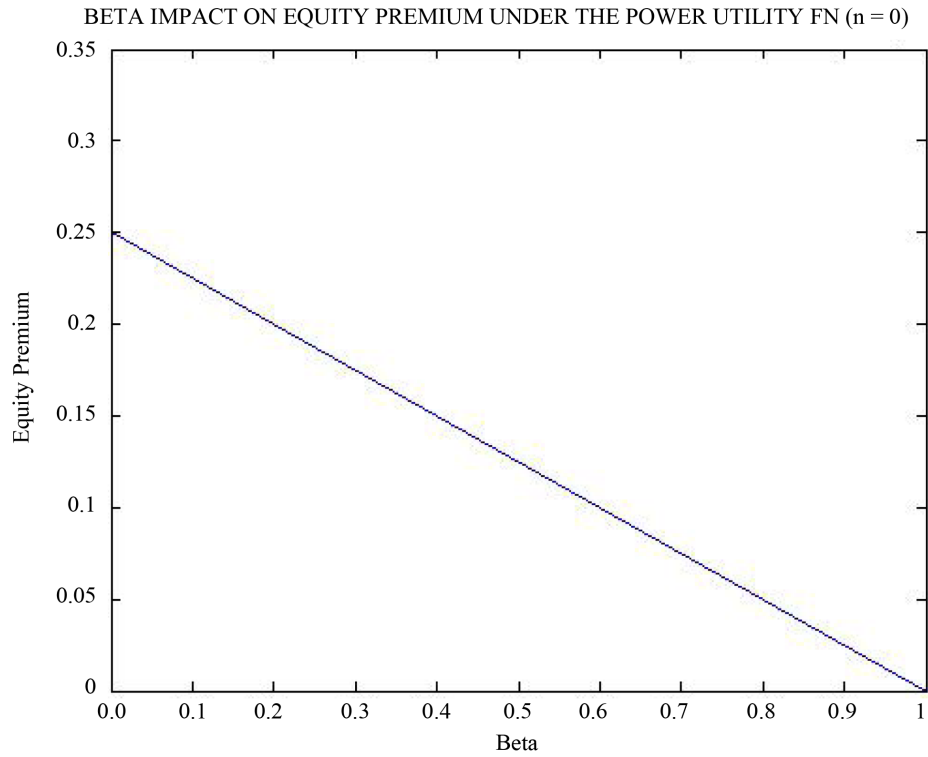BETA IMPACT ON EQUITY PREMIUM UNDER THE POWER UTILITY FN (n = 0)

Figure 4. Power utility beta effect under gamma distribution.

where $\phi_\delta = \frac{1}{2}\delta^2$ is the diffusive risk premium and

$$\phi_N = \lambda\left(1+(e-1)p\right)^n - \lambda\left(1-p+pe^{\frac{1}{2}}\right)^n - \lambda + \lambda\left(1-p+pe^{-\frac{1}{2}}\right)^n$$ is the rare-event premium.

Proof. For the square root utility function, the rare-event premium is given by

$$\phi_N = \lambda\mathbf{E}\left[\left(e^x-1\right)\left(1-e^{-\frac{1}{2}x}\right)\right] = \lambda\mathbf{E}\left[e^x - e^{\frac{1}{2}x} - 1 + e^{-\frac{1}{2}x}\right]$$

$$= \lambda\left[\mathbf{E}\left(e^x\right) - \mathbf{E}\left(e^{\frac{1}{2}x}\right) - \mathbf{E}(1) + \mathbf{E}\left(e^{-\frac{1}{2}x}\right)\right].$$

Since $x \sim B(n,p),$ we have that

$$\mathbf{E}\left[e^X\right] = (1-p+pe)^n = \left(1+(e-1)p\right)^n = m_X(1)$$

and

$$\mathbf{E}\left[e^{\frac{1}{2}X}\right] = \left(1-p+pe^{\frac{1}{2}}\right)^n = m_X\left(\frac{1}{2}\right).$$

Also

$$\mathbf{E}\left[e^{-\frac{1}{2}X}\right] = \left(1-p+pe^{-\frac{1}{2}}\right)^n = m_X\left(-\frac{1}{2}\right).$$

Thus our rare-event premium is

$$\lambda\left[\left(1+(e-1)p\right)^{n}-\left(1-p+pe^{\frac{1}{2}}\right)^{n}-1+\left(1-p+pe^{-\frac{1}{2}}\right)^{n}\right]$$

and therefore our equity premium is

$$\phi=\frac{1}{2}\delta^{2}+\lambda\left(1+(e-1)p\right)^{n}-\lambda\left(1-p+pe^{\frac{1}{2}}\right)^{n}-\lambda+\lambda\left(1-p+pe^{-\frac{1}{2}}\right)^{n}.$$ □

The equity premium is neither affected by the wealth value nor the time preference function and the diffusive risk premium is always positive.

Just as for the power utility function and normally distributed jump size, **Figure 5** suggest that, for $n=0$, the equity premium is almost zero when volatility is zero and fluctuates about a constant value when $n=55$.

Theorem 4. *In the production economy with jump diffusion under a vector x of jump sizes whose distribution follows a gamma, the investor's equilibrium equity premium with square root utility function* $U(r_{t})=\sqrt{r_{t}}, r_{t}>0,$ *is given by*

$$\phi=\frac{1}{2}\delta^{2}+\frac{\lambda}{\left(1-\frac{1}{\lambda}\right)^{k}}-\frac{\lambda}{\left(1-\frac{0.5}{\lambda}\right)^{k}}-\lambda+\frac{\lambda}{\left(1-\frac{(-0.5)}{\lambda}\right)^{k}}$$

where $\phi_{\delta}=\frac{1}{2}\delta^{2}$ is the diffusive risk premium and $\phi_{N}=\dfrac{\lambda}{\left(1-\frac{1}{\lambda}\right)^{k}}-\dfrac{\lambda}{\left(1-\frac{0.5}{\lambda}\right)^{k}}-\lambda+\dfrac{\lambda}{\left(1-\frac{(-0.5)}{\lambda}\right)^{k}}$ is the

rare-event premium.

Proof. For the square root utility function, the rare-event premium is given by

$$\phi_{N}=\lambda\mathbf{E}\left[\left(e^{x}-1\right)\left(1-e^{-\frac{1}{2}x}\right)\right]=\lambda\mathbf{E}\left[e^{x}-e^{\frac{1}{2}x}-1+e^{-\frac{1}{2}x}\right]$$

$$=\lambda\left[\mathbf{E}\left(e^{x}\right)-\mathbf{E}\left(e^{\frac{1}{2}x}\right)-\mathbf{E}\left(1\right)+\mathbf{E}\left(e^{-\frac{1}{2}x}\right)\right].$$

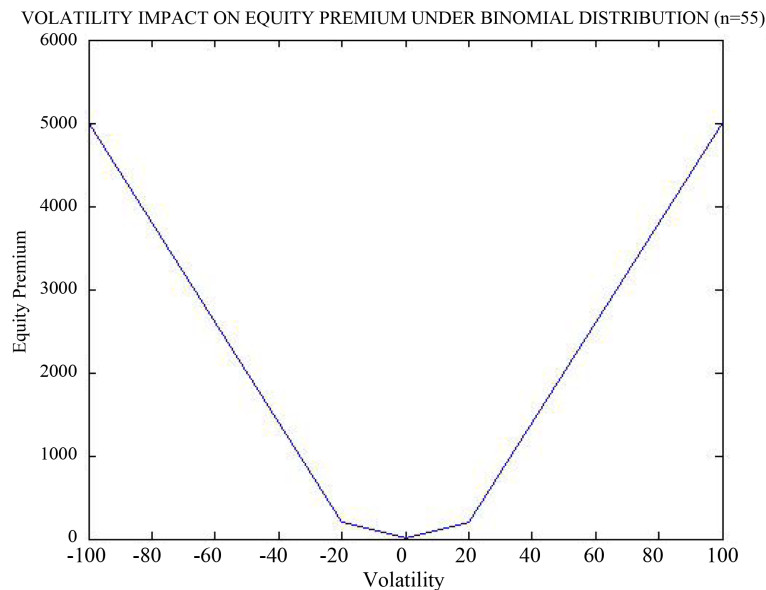**Figure 5.** Square root utility volatility effect under binomial distribution.

Now, since $x \sim G(\lambda, k)$,

$$E(e^x) = \frac{1}{\left(1 - \frac{1}{\lambda}\right)^k} = M_x(1)$$

$$E\left(e^{\frac{1}{2}x}\right) = \frac{1}{\left(1 - \frac{0.5}{\lambda}\right)^k} = M_x(0.5)$$

$$E\left(e^{-\frac{1}{2}x}\right) = \frac{1}{\left(1 - \frac{(-0.5)}{\lambda}\right)^k} = M_x(-0.5)$$

therefore

$$\phi_N = \frac{\lambda}{\left(1 - \frac{1}{\lambda}\right)^k} - \frac{\lambda}{\left(1 - \frac{0.5}{\lambda}\right)^k} - \lambda + \frac{\lambda}{\left(1 - \frac{(-0.5)}{\lambda}\right)^k}$$

and thus our equilibrium equity premium is

$$\phi = \frac{1}{2}\delta^2 + \frac{\lambda}{\left(1 - \frac{1}{\lambda}\right)^k} - \frac{\lambda}{\left(1 - \frac{0.5}{\lambda}\right)^k} - \lambda + \frac{\lambda}{\left(1 - \frac{(-0.5)}{\lambda}\right)^k}. \qquad \square$$

The equity premium is neither affected by the wealth value nor the time preference function and the diffusive risk premium is always positive. For $k = 0$, when volatility is zero, equity premium is zero. For $k = 55$, it decreases significantly as volatility approaches zero from either side (see **Figure 6**). This was the case also for the power utility function.

Theorem 5. *For the binomially distributed jump sizes, the investor's equilibrium equity premium with quadratic utility function* $U(r_t) = r_t - ar_t^2, a > 0$ *in the production economy with jump diffusion is given by*
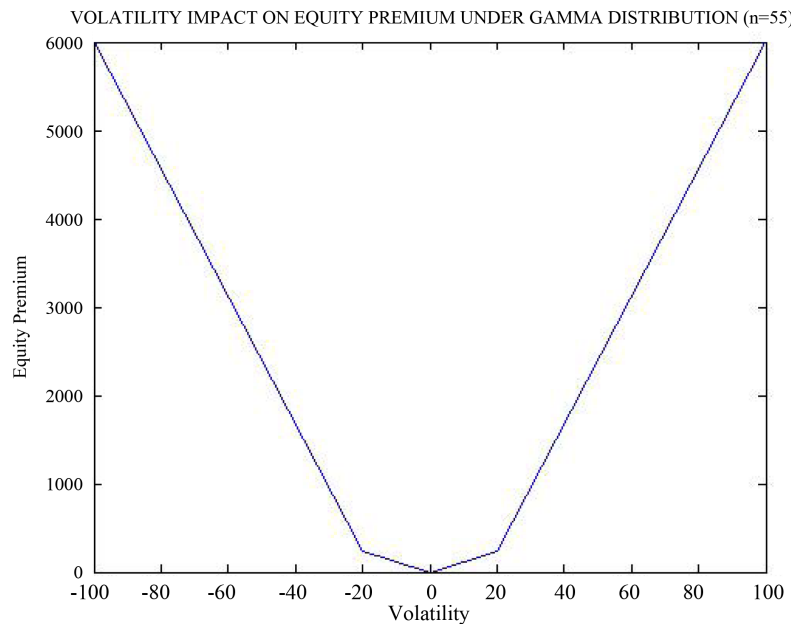

Figure 6. Square root utility volatility effect under gamma distribution.

$$\phi = \frac{2aV_t\delta^2}{1-2aV_t} + \lambda\left(1+(e-1)p\right)^n - \frac{\lambda\left(1+(e-1)p\right)^n}{1-2aV_t} + \frac{2a\lambda V_t\left(1-p+pe^2\right)^n}{1-2aV_t}$$

$$-\lambda + \frac{\lambda}{1-2aV_t} - \frac{2a\lambda V_t\left(1+(e-1)p\right)^n}{1-2aV_t}$$

where $\phi_\delta = \dfrac{2aV_t\delta^2}{1-2aV_t}$ is the diffusive risk premium and

$$\phi_N = \lambda\left(1+(e-1)p\right)^n - \frac{\lambda\left(1+(e-1)p\right)^n}{1-2aV_t} + \frac{2a\lambda V_t\left(1-p+pe^2\right)^n}{1-2aV_t} - \lambda + \frac{\lambda}{1-2aV_t} - \frac{2a\lambda V_t\left(1+(e-1)p\right)^n}{1-2aV_t}$$ is the

rare-event premium.

Proof. For the HARA Quadratic utility function,

$$\phi = \frac{2aV_t\delta^2}{1-2aV_t} + \lambda\mathbf{E}\left[\left(e^x-1\right)\left(1-\frac{\left(1-2aV_te^x\right)}{1-2aV_t}\right)\right]$$

where

$$\phi_N = \lambda\mathbf{E}\left[\left(e^x-1\right)\left(1-\frac{\left(1-2aV_te^x\right)}{1-2aV_t}\right)\right]$$

$$= \lambda\mathbf{E}\left[e^x - \frac{e^x-2aV_te^{2x}}{1-2aV_t} - 1 + \frac{1}{1-2aV_t} - \frac{2aV_te^x}{1-2aV_t}\right]$$

$$= \lambda\mathbf{E}\left[e^x - \frac{e^x}{1-2aV_t} + \frac{2aV_te^{2x}}{1-2aV_t} - 1 + \frac{1}{1-2aV_t} - \frac{2aV_te^x}{1-2aV_t}\right]$$

$$= \lambda\left[\mathbf{E}\left(e^x\right) - \frac{\mathbf{E}\left(e^x\right)}{1-2aV_t} + \frac{2aV_t\mathbf{E}\left(e^{2x}\right)}{1-2aV_t} - 1 + \frac{1}{1-2aV_t} - \frac{2aV_t\mathbf{E}\left(e^x\right)}{1-2aV_t}\right].$$

Now since $x \sim B(n,p)$, we have that

$$\mathbf{E}\left[e^X\right] = \left(1-p+pe\right)^n = \left(1+(e-1)p\right)^n = m_X(1)$$

and

$$\mathbf{E}\left[e^{2X}\right] = \left(1-p+pe^2\right)^n = m_X(2)$$

thus our rare-event premium is

$$\lambda\left[\left(1+(e-1)p\right)^n - \frac{\left(1+(e-1)p\right)^n}{1-2aV_t} + \frac{2aV_t\left(1-p+pe^2\right)^n}{1-2aV_t} - 1 + \frac{1}{1-2aV_t} - \frac{2aV_t\left(1+(e-1)p\right)^n}{1-2aV_t}\right]$$

which implies that our equity premium is

$$\phi = \frac{2aV_t\delta^2}{1-2aV_t} + \lambda\left(1+(e-1)p\right)^n - \frac{\lambda\left(1+(e-1)p\right)^n}{1-2aV_t} + \frac{2a\lambda V_t\left(1-p+pe^2\right)^n}{1-2aV_t}$$

$$-\lambda + \frac{\lambda}{1-2aV_t} - \frac{2a\lambda V_t\left(1+(e-1)p\right)^n}{1-2aV_t}.$$

□

It is not affected by the time preference function $y(t)$ but is affected by $V(t)$, the total wealth of the investor at any time t. **Figure 7** shows a constant equity premium regardless of how volatile the process

becomes. In terms of wealth value, the equity premium is zero whenever the wealth process is zero as shown in **Figure 8**. This result is consistent with the normal distribution of jump sizes and maybe attributed to the fact that, for a large sample size, a discrete process maybe used to approximate a continuous process.

Theorem 6. *For the gamma distribution of jump sizes, the investor's equilibrium equity premium with quadratic utility function* $U(r_t) = r_t - ar_t^2, a > 0$ *in the production economy with jump diffusion is given by*
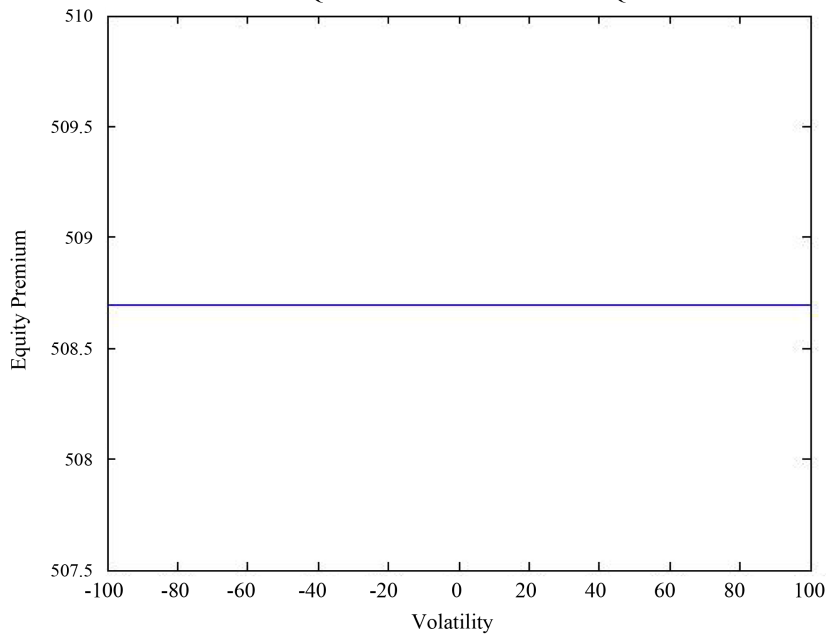
Figure 7. Quadratic utility volatility effect under binomial distribution.
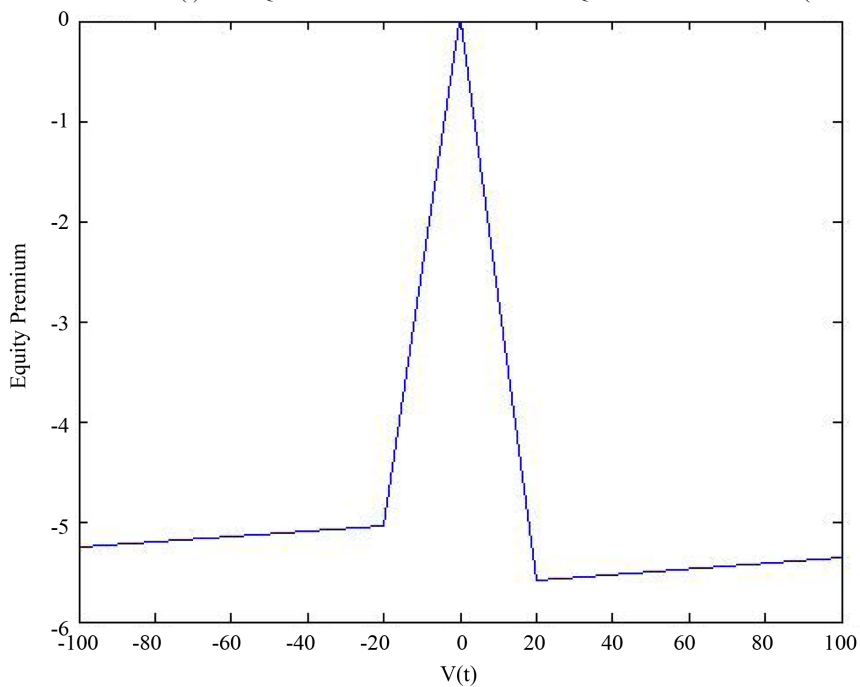

Figure 8. Quadratic utility wealth effect under binomial distribution.

$$\phi = \frac{2aV_t\delta^2}{1-2aV_t} + \frac{\lambda}{\left(1-\frac{1}{\lambda}\right)^k} - \frac{\lambda}{(1-2aV_t)\left(1-\frac{1}{\lambda}\right)^k} - \frac{2a\lambda V_t}{(1-2aV_t)\left(1-\frac{2}{\lambda}\right)^k}$$

$$-\lambda + \frac{\lambda}{1-2aV_t} - \frac{2a\lambda V_t}{(1-2aV_t)\left(1-\frac{1}{\lambda}\right)^k}$$

where $\phi_\delta = \dfrac{2aV_t\delta^2}{1-2aV_t}$ is the diffusive risk premium and

$$\phi_N = \frac{\lambda}{\left(1-\frac{1}{\lambda}\right)^k} - \frac{\lambda}{(1-2aV_t)\left(1-\frac{1}{\lambda}\right)^k} - \frac{2a\lambda V_t}{(1-2aV_t)\left(1-\frac{2}{\lambda}\right)^k} - \lambda + \frac{\lambda}{1-2aV_t} - \frac{2a\lambda V_t}{(1-2aV_t)\left(1-\frac{1}{\lambda}\right)^k} \quad \text{is the rare-event}$$

premium.

Proof. For the HARA Quadratic utility function,

$$\phi = \frac{2aV_t\delta^2}{1-2aV_t} + \lambda\mathbf{E}\left[(e^x-1)\left(1-\frac{(1-2aV_te^x)}{1-2aV_t}\right)\right]$$

where

$$\phi_N = \lambda\mathbf{E}\left[(e^x-1)\left(1-\frac{(1-2aV_te^x)}{1-2aV_t}\right)\right]$$

$$= \lambda\mathbf{E}\left[e^x - \frac{e^x-2aV_te^{2x}}{1-2aV_t} - 1 + \frac{1}{1-2aV_t} - \frac{2aV_te^x}{1-2aV_t}\right]$$

$$= \lambda\mathbf{E}\left[e^x - \frac{e^x}{1-2aV_t} - \frac{2aV_te^{2x}}{1-2aV_t} - 1 + \frac{1}{1-2aV_t} - \frac{2aV_te^x}{1-2aV_t}\right]$$

$$= \lambda\left[\mathbf{E}(e^x) - \frac{\mathbf{E}(e^x)}{1-2aV_t} - \frac{2aV_t\mathbf{E}(e^{2x})}{1-2aV_t} - 1 + \frac{1}{1-2aV_t} - \frac{2aV_t\mathbf{E}(e^x)}{1-2aV_t}\right].$$

Now since $x \sim G(\lambda, k)$, we have that

$$\mathbf{E}(e^x) = \frac{1}{\left(1-\frac{1}{\lambda}\right)^k} = M_x(1)$$

$$\mathbf{E}(e^{2x}) = \frac{1}{\left(1-\frac{2}{\lambda}\right)^k} = M_x(2)$$

thus

$$\phi_N = \lambda\left[\frac{1}{\left(1-\frac{1}{\lambda}\right)^k} - \frac{\frac{1}{\left(1-\frac{1}{\lambda}\right)^k}}{1-2aV_t} - \frac{2aV_t\frac{1}{\left(1-\frac{2}{\lambda}\right)^k}}{1-2aV_t} - 1 + \frac{1}{1-2aV_t} - \frac{2aV_t\frac{1}{\left(1-\frac{1}{\lambda}\right)^k}}{1-2aV_t}\right]$$

which is just

$$\frac{\lambda}{\left(1-\frac{1}{\lambda}\right)^{k}} - \frac{\lambda}{\left(1-2aV_{t}\right)\left(1-\frac{1}{\lambda}\right)^{k}} - \frac{2a\lambda V_{t}}{\left(1-2aV_{t}\right)\left(1-\frac{2}{\lambda}\right)^{k}} - \lambda + \frac{\lambda}{1-2aV_{t}} - \frac{2a\lambda V_{t}}{\left(1-2aV_{t}\right)\left(1-\frac{1}{\lambda}\right)^{k}}.$$

So that our equilibrium equity premium is now

$$\phi = \frac{2aV_{t}\delta^{2}}{1-2aV_{t}} + \frac{\lambda}{\left(1-\frac{1}{\lambda}\right)^{k}} - \frac{\lambda}{\left(1-2aV_{t}\right)\left(1-\frac{1}{\lambda}\right)^{k}} - \frac{2a\lambda V_{t}}{\left(1-2aV_{t}\right)\left(1-\frac{2}{\lambda}\right)^{k}}$$

$$- \lambda + \frac{\lambda}{1-2aV_{t}} - \frac{2a\lambda V_{t}}{\left(1-2aV_{t}\right)\left(1-\frac{1}{\lambda}\right)^{k}}.$$

□

It is not affected by the time preference function $y(t)$, but is affected by $V(t)$, the total wealth of the investor at any time t. As evident in **Figure 9**, although for $k=0$ the equity premium is negative, it rises significantly as the wealth value process moves from negative to zero and becomes zero when the wealth process is zero. The equity premium decreases significantly when the investor's wealth is in the range 0 to 20 and begins to rise again. For $k=55$, the wealth process V_{t} affects the equity premium in the same way.

4. Conclusions

In conclusion, the optimal consumption of the investor is affected by the deterministic time preference function $y(t)$ but it has no effect on the diffusive and rare-events premia. For $k=0$, the equity premium is almost zero when the volatility is zero. However, it is non zero for $k=55$ even if it is symmetrical about zero premium. In fact, it decreases significantly as volatility approaches zero from either side. The equity premium for the quadratic utility function is affected by V_{t} the total wealth of an investor at time t. When $V_{t}=0$, the equity premium is zero. However, for $k=55$, it is constant regardless of how volatile the process becomes.
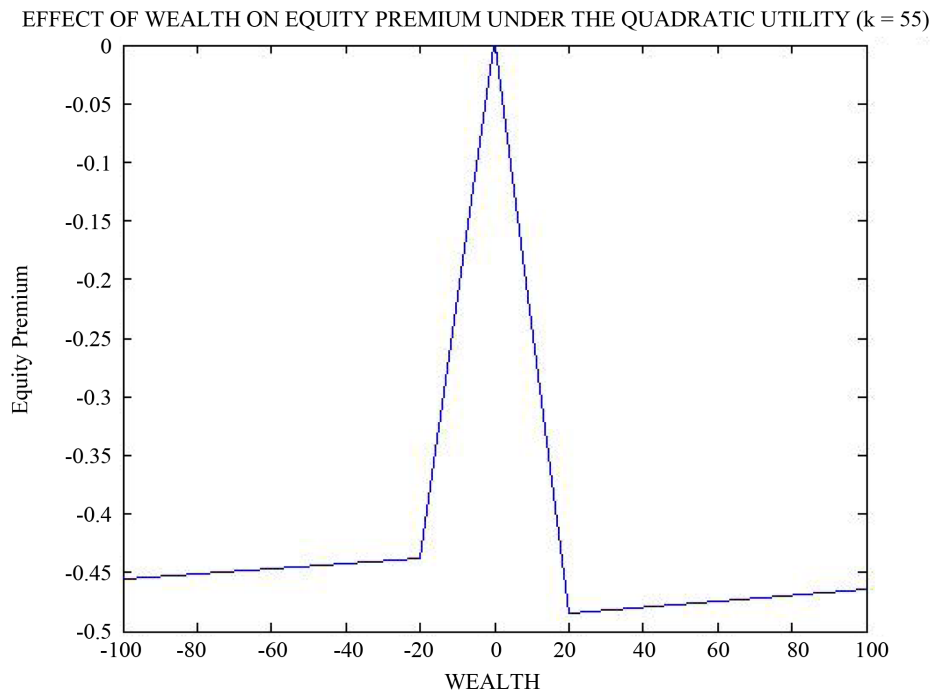

EFFECT OF WEALTH ON EQUITY PREMIUM UNDER THE QUADRATIC UTILITY (k = 55)

Figure 9. Quadratic utility wealth effect under gamma distribution.

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